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Series expansions for the percolation probability of a generalized Domany–Kinzel cellular automaton

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Abstract. Series expansions have been derived for the percolation probability of a generalized Domany–Kinzel cellular automaton with two equivalent absorbing states. The analysis of the series generally yields estimates of the critical exponent $\beta = 1.00 \pm 0.05$, consistent with earlier Monte Carlo studies thus confirming that the model belongs to the same universality class as branching annihilating random walks with an even number of offspring. There is evidence to suggest that when the probability of spreading from two active sites becomes small a new critical behaviour emerges.

1. Introduction

Models exhibiting a continuous phase transition from an 'active' steady state to an absorbing 'inactive' state are encountered in a wide variety of problems such as fluid flow in porous media, chemical reactions, population dynamics, catalysis, epidemics, forest fires, biological and even galactic evolution. By far the most ubiquitous of these classes is that of directed percolation (DP). It is by now generally accepted that DP is the generic universality class for non-equilibrium models with such active-to-inactive phase transitions. A recent review of many models in this class can be found in [1]. The only major exception to the DP rule is a set of models with an additional local conservation law and/or symmetry among different absorbing states. Among the first such models were a probabilistic cellular automaton [2, 3] and a kinetic Ising model [4]. Branching annihilating random walks with an *even* number of offspring, where the number of particles is conserved locally modulo 2, also belongs to this universality class [5–8], which I shall refer to as the *parity conserving* (PC) universality class. Recently many new models have been reported as belonging to this new universality class [9–15].

As demonstrated by Domany and Kinzel [16], DP on the square lattice can be seen as a one-dimensional stochastic cellular automaton in which the preferred direction t is time. DP is thus a model for a simple branching process in which a site x occupied at time t may give rise to zero or one offspring on each of the sites $x \pm 1$ at time $t + 1$. The evolution of the model is determined by the conditional probabilities $W(\sigma_x|\sigma_l, \sigma_r)$ of finding the site (x, t) in state σ_x given that the sites $(x - 1, t - 1)$ and $(x + 1, t - 1)$ were in states σ_l and σ_r, respectively, with $\sigma_i = 1$ if site i is occupied and 0 otherwise. One has a free hand in choosing these probabilities as long as one respects conservation of probability, $W(1|\sigma_l, \sigma_r) = 1 - W(0|\sigma_l, \sigma_r)$, and the condition $W(1|0, 0) = 0$ which ensures that the

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Table 1. Evolution rules for the generalized Domany–Kinzel model with two equivalent absorbing states.

σ_l, σ_r	AA	AI_1	I_1A	AI_2	I_2A	I_1I_1	I_2I_2	I_1I_2	I_2I_1
$W(A\|\sigma_l, \sigma_r)$	p_2	p_1	p_1	p_1	p_1	0	0	1	1
$W(I_1\|\sigma_l, \sigma_r)$	$(1-p_2)/2$	$1-p_1$	$1-p_1$	0	0	1	0	0	0
$W(I_2\|\sigma_l, \sigma_r)$	$(1-p_2)/2$	0	0	$1-p_1$	$1-p_1$	0	1	0	0

process has an absorbing state. Bond and site percolation correspond to the particular choices

$$W(0|\sigma_l, \sigma_r) = (1-p)^{\sigma_l+\sigma_r} \qquad \text{(bond)} \tag{1}$$

$$W(0|\sigma_l, \sigma_r) = (1-p)^{\sigma_l+\sigma_r-\sigma_l\sigma_r} \qquad \text{(site)}. \tag{2}$$

Recently, Hinrichsen [15] studied a generalized version of the Domany–Kinzel model in which there are n equivalent absorbing states. For $n = 1$ one recovers the usual Domany–Kinzel model, while for $n = 2$ the model belongs to the PC universality class provided the symmetry between absorbing states is preserved. In the case $n = 2$ each site can be in one of three states: there is one active state A and two equivalent inactive states I_1 and I_2. The evolution is governed by the conditional probabilities listed in table 1. In this paper I shall always be looking at situations in which p_2 is a simple function (polynomial) of $p_1 = p$, e.g. the cases $p_2 = p$ and $p_2 = 2p - p^2$, which are the analogues of site and bond DP, respectively. The behaviour of the Domany–Kinzel model is controlled by the branching probability p. When p is smaller than a critical value p_c the branching process always dies out, while for $p > p_c$ there is a non-zero probability $P(p)$ it will survive indefinitely. At p_c the survival probability vanishes as a power law,

$$P(p) \propto (p - p_c)^{\beta} \qquad p \to p_c^+. \tag{3}$$

In the case of ordinary DP the critical exponent β is known to a very high degree of accuracy. The most accurate estimate is that of [17] $\beta = 0.276\,49(4)$, where the number in parentheses indicate the error in the last digit. The estimates of β for the PC universality class differ from the DP value and generally the numerical evidence suggests $\beta = 0.95(5)$, where the rather large error reflects the inherent difficulty in obtaining accurate estimates for β and the often rather large discrepancy among the various studies. Here I shall briefly review the various estimates. There are two different ways of defining and measuring the exponent β, the first of which was given above. The second is through measuring how the steady-state concentration of active sites vanishes at p_c. In the first case one studies the ultimate survival probability (as a function of p) of a system with initially just one or two active sites while in the second case one typically starts with a large lattice with all sites initially active and then measures the ultimate (large t limit) concentration of active sites. For DP there is ample numerical and theoretical evidence that the two exponents are identical, however, this need not necessarily be the case [18, 19]. Estimates for β based on the first method are 0.94(6) [3], 0.93(5) [6], and 0.97(8) [8], while the estimates from the second method include 0.922(5) [7], 0.88(4) [10], 0.88(3) [11], and 0.90(5) and 0.93(5) [15]. Given the present accuracy of these estimates there is no compelling reason to believe that the two β's are different. In an attempt to obtain more accurate estimates for β I have calculated a series expansion for the percolation probability of the generalized Domany–Kinzel model.

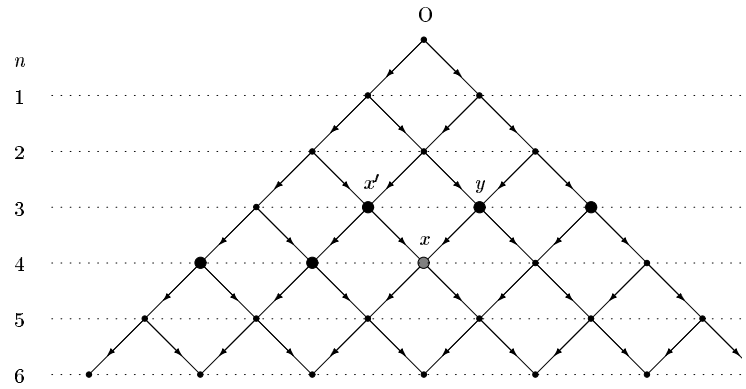

Figure 1. The directed square lattice with orientation given by the arrows. The sites marked by large filled circles indicate the position of the boundary line prior to moving it from the centre site x' to the shaded site at x.

2. Series expansion technique

Consider the calculation of the series expansion of the percolation probability for directed percolation [20, 21] on a square lattice oriented as in figure 1. Clusters growing from a single initial site can only reach the sites shown in figure 1 below the origin O. This naturally leads to a finite-lattice approximation to P, namely the probability P_n that the origin is connected to at least one site in the nth row. P_n is a polynomial in the variable $q = 1 - p$ (this turns out to be the natural high-density variable) with integer coefficients, and the coefficients of q^k are identical to those of $P(q)$ for $k \leqslant n + 1$. For the generalized Domany–Kinzel model the calculation of the series expansion for the percolation probability is essentially unchanged, and I will therefore only briefly describe the method. $P_n(q)$ is calculated as $1 - \tilde{P}_n(q)$, where $\tilde{P}_n(q)$ is the probability that no paths lead to level n, while starting from a single active site with all other sites in the initial row being in either state I_1 or state I_2. As stated earlier the inactive states are equivalent and from now on I shall assume that I_1 is the 'background' state. If no paths lead to level n all sites on level n (and onward) must be in state I_1, i.e. the system has reached an absorbing state. As is the case for DP $P_n(q)$ yields the first $n + 1$ terms of the series expansion for $P(q)$. $\tilde{P}_n(q)$ can be calculated by summing over all configurations on the lattice in figure 1 with the origin in the active state and all states on level n (and outside those shown in the figure) in state I_1. Each configuration carries a weight given by the product over all the local weights of down-pointing triangles, such as that formed by the three sites (x, x', y) which contribute the weight $W(\sigma_x|\sigma_{x'}, \sigma_y)$. The sum over all configurations can be performed by moving a boundary line through the lattice. At any given stage this line cuts through a number of, say m, lattice sites thus leading to a total of 3^m possible configurations along this line. For each configuration along the boundary line one maintains a (truncated) polynomial which equals the sum of the product of weights over all possible states on the side of the boundary already traversed. The boundary is moved through the lattice one site at a time. In figure 1 we show how the boundary is moved in order to pick up the weight associated with a given triangle at position x along the boundary line. Let $S_x = (\sigma_1, \ldots, \sigma_x, \ldots, \sigma_m)$, be the configuration of sites along the boundary with state σ_x at position x, where $\sigma_i = 0$, 1, or 2 corresponding to, e.g. the state being either I_1, I_2, or A, respectively. Then in moving the boundary from x' to x, from the top left to the bottom of the triangle formed by the sites

(x, x', y), the polynomials associated with these configurations are updated as

$$P(S_x) = \sum_{\sigma_{x'}} W(\sigma_x|\sigma_{x'}, \sigma_y) P(S_{x'}). \tag{4}$$

As is the case for DP the major limitation is available memory and likewise this limitation can be partially overcome by using a 'pivoting' algorithm [20, 21]. In this approach one cuts the lattice in half with a line of sites, fixed in a particular state, which is used as a pivot-line for the moving boundary. One now has to calculate the weight for each configuration of fixed sites and obtain the final result by summing over all possible configurations. In this manner the memory requirements grows as $3^{n/2}$ rather than 3^n. The downside is an increase in computer time and a more complicated algorithm. In this case the line of fixed sites starts in the centre at level n (recall that all sites on level $n+1$ are in state I_1) and is parallel to the left edge of the finite lattice and terminates just before the right edge.

I calculated the percolation probability series for three different cases. The first case was for the general two-parameter model with the weights of table 1 using the variables $q_1 = 1 - p_1$ and $q_2 = (1 - p_2)/2$. The need to retain a two-parameter expansion is costly computationally. Longer series can be generated if one looks at specific choices for p_2. I have studied the special cases $p_2 = p_1 = p$ and $p_1 = p, p_2 = 2p - p^2$, which are the analogues of site and bond DP, respectively. In both these cases I used the expansion variable $q = (1 - p)/2$. The choice of expansion variables ensures that the coefficients appearing in P_n are integers. In the general case I calculated $P_n(q_1, q_2)$ up to $n = 20$, while I was able to extend the calculations to $n = 23$ and 24 for the bond and site cases, respectively.

For bond DP on the square lattice Baxter and Guttmann [20] demonstrated that the series for $P(q)$ can be extended considerably by determining correction terms to $P_n(q)$:

$$P_n - P_{n+1} = q^{n+2} \sum_{r \geqslant 0} q^r d_{n,r}. \tag{5}$$

We shall call $d_{n,r}$ the rth correction term. Obviously if one can find formulae for $d_{n,r}$ for all $r \leqslant k$ then one can use the series coefficients of $P_n(q)$ to extend the series for $P(q)$ to order $n + k + 2$ since

$$a_{n+2+j} = a_{n,n+2+j} - \sum_{i=0}^{j} d_{n+k-i,i} \tag{6}$$

for all $j \leqslant k$, where a_i and $a_{n,i}$ are the coefficients in $P(q)$ and $P_n(q)$, respectively. For this case the first correction term was conjectured to be the Catalan numbers [20], $d_{n,0} = C_n = (2n)!/(n!(n+1)!)$. This conjecture has since been proved [22, 23]. Baxter and Guttmann also found that the higher-order correction terms can be expressed as linear functions of $d_{n,0}$. For the generalized Domany–Kinzel model I find that the first correction term is given by a quite simple recurrence relation which is readily identified using the Gfun package [24]. In the two-variable case I looked at the generalizations of the site and bond cases, i.e. $q_1 = (1 - p_1)/2$ with $q_2 = zq_1$ and $q_2 = zq_1^2$, respectively, where z is a constant. The first correction term for the bond case is very simple and independent of z, $d_{n,0} = 2^{n+1}C_n$. The factors 2^{n+1} merely arise because the expansion variable is $q_1 = (1 - p_1)/2$ rather than $1 - p$ as for bond DP. However, for the site case $d_{n,0}$ depends on z and is given by the recurrence relation,

$$\begin{aligned} d_{n+3,0} = \{&[2(3z^3 + 8z^2 + 13z - 1)n^2 + (39z^3 + 124z^2 + 170z - 14)n \\ &+3(21z^3 + 79z^2 + 92z - 8)]d_{n+2,0} - 3[2(z^4 - z^3 - 9z^2 + 16z - 4)n^2 \\ &+(9z^4 - 13z^3 - 66z^2 + 132z - 40)n + (10z^4 - 13z^3 - 56z^2 + 132z - 48)] \end{aligned}$$

$$\times d_{n+1,0} + (z^5 - 10z^4 + 40z^3 - 80z^2 + 80z - 32)(2n^2 + 5n + 3)d_{n,0}\}$$
$$/[z(z+1)(2n^2 + 17n + 36)]$$
$$d_{2,0} = 4z^2 + 48z + 80 \qquad d_{1,0} = 4z + 16 \qquad d_{0,0} = 4. \tag{7}$$

Thus for any value of z one can derive a series correct to order 22.

In the two special cases the extension procedure was carried further and formulae were found for the first three correction terms in the site case and the first seven correction terms in the bond case. The percolation probability series for the site case was thus extended to order 29 while the series was extended to order 32 in the bond case. The procedure for finding the formulae for the higher-order correction terms is very similar to the DP case [20, 21], i.e. the higher-order correction terms are expressed as linear functions of the first correction term. Readers interested in the details can contact the author for further information.

3. Series analysis and results

The series were analysed using Dlog–Padé approximants (see [25] for a review), which yields estimates for β and q_c. Here it suffices to say that a $[L, M]$ Dlog–Padé approximant to a function f is formed by approximating the logarithmic derivative of f by the ratio of two polynomials

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln f(x) = \frac{P_L(x)}{Q_M(x)}. \tag{8}$$

Q_M and P_L are polynomials of order M and L, respectively, whose coefficients are chosen such that the series expansion of P_L/Q_M agree with the first $L+M+1$ terms of $(\mathrm{d}/\mathrm{d}x) \ln f$. The possible singularities of the series appear as the zeros of the polynomial Q_M and the associated critical exponent is estimated from the residue. The physical singularity should appear as the first zero on the positive real axis.

Before analysing the bond and site series a change of variable, $q = 2q_1$, was performed so that $q = 1 - p_1$. The estimates obtained from the Dlog–Padé analysis are listed in tables 2 and 3, respectively. The analysis of the site series yields estimates of β close to 1. However, there is evidence that as the order of the approximants increase the estimates tend to drift lower. The wide majority of approximants are consistent with the estimates

Table 2. Estimates of q_c and β from Dlog–Padé approximants to the percolation probability series for the *site* problem

	$[M-2, M]$		$[M-1, M]$		$[M, M]$		$[M+1, M]$		$[M+2, M]$	
M	q_c	β	q_c	β	q_c	β	q_c	β	q_c	β
6	0.431 150	0.9740	0.432 205	1.0002	0.432 445	1.0068	0.432 443	1.0067	0.431 777	0.9964
7	0.433 229	1.0323	0.432 930	1.0217	0.433 480	1.0449	0.432 779	1.0160	0.434 562	1.1308
8	0.433 068	1.0267	0.433 041	1.0257	0.433 109	1.0284	0.433 235	1.0341	0.433 068	1.0262
9	0.433 061	1.0264	0.432 834	1.0199	0.433 167	1.0309	0.433 616	1.0523	0.432 938	1.0205
10	0.433 099	1.0279	0.433 103	1.0280	0.433 083	1.0272	0.432 898	1.0187	0.432 923	1.0198
11	0.433 100	1.0279	0.433 106	1.0281	0.432 945	1.0209	0.432 915	1.0195	0.432 926	1.0200
12	0.433 098	1.0278	0.432 767	1.0113	0.432 798	1.0132	0.433 025	1.0239	0.432 887	1.0182
13	0.432 801	1.0134	0.432 770	1.0115	0.432 378	0.9807	0.432 156	0.9567	0.432 887	1.0182
14	0.432 598	0.9999	0.432 092	0.9488	0.432 378	0.9807				
15	0.432 598	0.9999								

Table 3. Estimates of q_c and β from Dlog–Padé approximants to the percolation probability series for the *bond* problem

	$[M-2, M]$		$[M-1, M]$		$[M, M]$		$[M+1, M]$		$[M+2, M]$	
M	q_c	β	q_c	β	q_c	β	q_c	β	q_c	β
6	0.441 489	0.7592	0.440 786	0.7470	0.441 371	0.7570	0.428 888	0.6995	0.446 679	0.8869
7	0.441 362	0.7568	0.441 011	0.7503	0.453 068	1.2101	0.450 513	1.0519	0.441 808	0.7774
8	0.447 015	0.9010	0.450 690	1.0618	0.452 405	1.1655	0.455 304	1.4070	0.454 099	1.2914
9	0.440 350	0.7766	0.455 868	1.4691	0.454 379	1.3171	0.456 860	1.5801	0.452 262	1.1462
10	0.453 736	1.2597	—	—	0.452 999	1.2026	0.450 938	1.0533	0.451 405	1.0857
11	0.452 351	1.1531	0.451 655	1.1035	0.451 472	1.0905	0.451 825	1.1171	0.452 985	1.2291
12	0.451 446	1.0886	0.451 570	1.0976	0.454 368	1.4356	0.451 867	1.1204	0.451 627	1.1016
13	0.451 964	1.1276	0.451 622	1.1013	0.451 652	1.1035	0.451 611	1.1003	0.451 674	1.1048
14	0.451 650	1.1033	0.451 636	1.1023	0.691 897	0.6134	0.451 622	1.1012	0.451 643	1.1028
15	0.451 412	1.0811	0.451 666	1.1045	0.451 718	1.1080	0.451 622	1.1012		
16	0.451 706	1.1072	0.451 666	1.1045						

$q_c = 0.4327(4)$ and $\beta = 1.00(3)$. This is also the special case studied in [15] with the result $p_c = 0.5673(5)$ and $\beta = 0.90(5)$. While the estimates for the critical point are in excellent agreement the estimates for β do not overlap. The β measured by Hinrichsen is from Monte Carlo simulations of the steady-state concentration of active sites, a method which is often quite inaccurate due to finite-size corrections and convergence problems stemming from critical slowing down. Despite the fact that the two β's could be different, as discussed earlier, they are probably identical. The slight difference observed here could easily be due to underestimation of the error bars. For the bond case the wide variety of approximants favour the estimates $q_c = 0.4516(3)$ and $\beta = 1.10(3)$. The estimate for β is quite different from other studies and not really consistent with the estimates for the site case or the studies quoted in the introduction. If we check with the general bond case (table 4) we note that this special case, corresponding to $z = 5$, stands out as yielding a particularly large estimate for β.

In the two-variable case the series were calculated for $q_1 = (1-p_1)/2$ with $q_2 = zq_1/10$ and $q_2 = zq_1^2/5$, respectively, where z is a positive integer. Before analysing the series a change of variable, $q = 2q_1$, was performed. Table 4 lists estimates of q_c and β for various values of z. The estimates were obtained as an average over most of the approximants with $|M-L| \leqslant 2$ and $M+L \geqslant 18$. The quoted errors were calculated as one standard deviation among the approximants used in obtaining the estimates. As one would expect, given the significantly shorter series, the estimates are not as well converged as in the special cases. Generally the estimates are consistent with $\beta \simeq 1$. Unfortunately most of the estimates are marred by quite large error bars. This is especially true for large z; furthermore, note that the series for the site case generally appear to yield estimates with smaller error bars. However, given that one expects β to be the same for all z and in both cases, all one can say confidently from this analysis is that $\beta = 1.0 \pm 0.1$, which obviously is fully consistent with the estimates given above and those from other studies quoted in the introduction. It is, however, worth noting that in many cases for intermediate values of z the error bars are quite small, and in particular the site case seems to favour a value of β close to 1. In both cases (especially the site case) one observes that the β-estimates seems to decrease systematically with increasing z, but the error bars are too large to determine whether this a true effect or just a coincidence.

Table 4. Estimates of q_c and β from Dlog–Padé approximants to the percolation probability series for various values of z for the generalized site and bond problems.

	Site		Bond	
z	q_c	β	q_c	β
1	0.467 50(27)	0.657 5(83)	0.467 92(14)	0.6432(43)
2	0.456 36(28)	0.678 8(29)	0.456 7(12)	0.6597(89)
3	—	—	—	—
4	0.454 3(24)	1.19(21)	—	—
5	0.449 6(12)	1.111(98)	0.4524(10)	1.167(81)
6	0.445 67(23)	1.084(15)	0.447 82(24)	1.071(12)
7	0.442 00(10)	1.059 6(61)	0.444 28(33)	1.043(16)
8	0.438 63(25)	1.038(14)	0.440 7(16)	1.008(72)
9	0.435 81(30)	1.034(15)	0.437 8(17)	0.993(75)
10	0.433 102 5(25)	1.028 01(10)	0.436 2(10)	1.023(43)
11	0.431 27(24)	1.036 5(33)	0.433 9(20)	1.028(92)
12	0.427 30(39)	0.961(25)	0.427 6(50)	0.90(12)
13	0.424 96(20)	0.957(11)	0.426 0(43)	0.90(12)
14	0.422 62(47)	0.944(26)	0.430 4(53)	1.14(24)
15	0.419 99(75)	0.910(39)	0.427 9(38)	1.15(14)
16	0.4187(17)	0.940(81)	0.427 5(44)	1.16(18)
17	0.415 27(90)	0.865(41)	0.425 1(28)	1.12(11)
18	0.414 31(63)	0.901(26)	0.423 1(25)	1.110(97)
19	0.410 9(17)	0.830(73)	0.420 0(50)	1.07(15)

Table 5. Estimates of q_c and β from Dlog–Padé approximants to the percolation probability series for the generalized *site* problem with $z = 1$ and 2.

	$[N-2, N]$		$[N-1, N]$		$[N, N]$		$[N+1, N]$		$[N+2, N]$	
N	q_c	β	q_c	β	q_c	β	q_c	β	q_c	β
$z = 1$										
8	0.462 991	0.4924	0.466 172	0.6141	0.466 335	0.6197	0.467 795	0.6660	0.467 437	0.6551
9	0.470 288	0.7252	0.467 944	0.6705	0.467 164	0.6463	0.467 542	0.6584	0.467 282	0.6507
10	0.464 157	0.5310	0.467 591	0.6600	0.467 871	0.6690	0.466 374	0.6290	0.467 281	0.6507
11	0.467 768	0.6656	0.467 043	0.6448	0.467 870	0.6690				
$z = 2$										
8	0.456 243	0.6770	0.456 405	0.6790	0.456 157	0.6754	0.456 369	0.6783	0.456 401	0.6787
9	0.456 331	0.6779	0.456 717	0.6832	0.456 401	0.6787	0.456 368	0.6782	—	—
10	0.456 465	0.6797	0.456 639	0.6821	0.459 227	0.7410	—	—	—	—
11	0.455 760	0.6729	—	—	0.459 224	0.7409				

The only exemptions to the general behaviour described above are the cases $z = 1$ and 2 where significantly smaller estimates for β occur. For the value $z = 3$ (and $z = 4$ in the bond case) the series are so ill behaved that no meaningful estimates could be obtained. Table 5 lists the estimates obtained from Dlog–Padé approximants to the generalized site problem at the values $z = 1$ and 2. As can be seen the estimates for $z = 1$ are quite stable and favour a value of $\beta = 0.66(1)$, though there are quite a few approximants yielding very different estimates. The situation for $z = 2$ (and for the bond case) is similar, though in this case the spread is greater and many approximants yield no estimate at all. This raises the intriguing possibility that for low values of z a different critical behaviour occurs. The

fact that no estimates could be obtained for $z = 3$ supports the view that something novel happens as z is decreased. However, given the relatively short series and that the estimates are quite scattered no firm conclusion can be reached at present. It should be mentioned that for $z = 0$ the series show that $P(q) = (1 - 2q)/(1 - q)^2$, so in this case there is a critical point at $q_c = \frac{1}{2}$ with exponent $\beta = 1$.

4. Summary and discussion

Series have been derived for the percolation probability of a generalized Domany–Kinzel cellular automaton with two equivalent absorbing states. Analysis of the series clearly demonstrates that this model belongs to the parity conserving universality class of branching annihilating random walks with an even number of offspring. Contrary to what one might have hoped, the estimates obtained for the critical exponents β are not very accurate. Based on the analysis of the site case I estimate that $\beta = 1.00(5)$. While this estimate is a little higher than those generally obtained for this universality class, it is not inconsistent with earlier studies. An unresolved inconsistency is that the analysis of the bond case favours the higher estimate $\beta = 1.10(3)$. More interesting is the possibility that for low values of z a new critical behaviour occurs. Further study of this model is clearly warranted.

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References

[1] Marro J and Dickman R 1996 *Nonequilibrium Phase Transitions in Lattice Models* (Cambridge: Cambridge University Press)
[2] Grassberger P, Krause F and von der Twer T 1984 *J. Phys. A: Math. Gen.* **17** L105
[3] Grassberger P 1989 *J. Phys. A: Math. Gen.* **22** L1103
[4] Droz M, Rácz Z and Schmidt J 1989 *Phys. Rev.* A **39** 2141
[5] Takayasu H and Tretyakov A Y 1992 *Phys. Rev. Lett.* **68** 3060
[6] Jensen I 1994 *Phys. Rev.* E **50** 3623
[7] Zhong D and ben Avraham D 1995 *Phys. Lett.* A **209** 333
[8] Inui N 1995 *J. Phys. Soc. Japan* **64** 2266
[9] Menyhárd N 1994 *J. Phys. A: Math. Gen.* **27** 6139
[10] Menyhárd N and Ódor G 1996 *J. Phys. A: Math. Gen.* **29** 7739
[11] Kim M H and Park H 1994 *Phys. Rev. Lett.* **73** 2579
[12] Park H, Kim M H and Park H 1995 *Phys. Rev.* E **52** 5664
[13] Park H and Park H 1995 *Physica* **221A** 97
[14] Bassler K E and Browne D A 1996 *Phys. Rev. Lett.* **77** 4094
[15] Hinrichsen H 1997 *Phys. Rev.* E **55** 219
[16] Domany E and Kinzel W 1984 *Phys. Rev. Lett.* **53** 311
[17] Jensen I 1996 *J. Phys. A: Math. Gen.* **29** 7013
[18] Jensen I and Dickman R 1993 *Phys. Rev.* E **48** 1710
[19] Mendes J F F, Dickman R, Henkel M and Marquez M C 1994 *J. Phys. A: Math. Gen.* **27** 3019
[20] Baxter R and Guttmann A J 1988 *J. Phys. A: Math. Gen.* **21** 3193
[21] Jensen I and Guttmann A J 1995 *J. Phys. A: Math. Gen.* **28** 4813
[22] Bousquet-Mélou M 1996 *Eur. J. Comb.* **17** 343
[23] Inui N and Katori M 1996 *J. Phys. A: Math. Gen.* **29** 4347
[24] Salvy B and Zimmermann P 1994 *ACM Trans. Math. Software* **20** 162
[25] Guttmann A J 1989 *Phase Transitions and Critical Phenomena* vol 13, ed C Domb and J L Lebowitz (London: Academic) pp 1–234